

THE COMBINATORICS OF OCCAM'S RAZOR

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ABSTRACT. Occam's Razor tells us to pick the simplest model that fits our observations. In order to make sense of his process mathematically, we interpret it in the context of posets of functions. Our approach leads to some unusual new combinatorial problems concerning functions between finite sets. The same ideas are used to define a nicely behaved and apparently unknown analogue of the rank of a group. We also make a construction that associates with each group an infinite sequence of numbers called its fusion sequence. The first term in this sequence is determined by the rank of the group and we provide examples of subsequent terms that suggest a subtle relationship between these numbers and the structure of the group.

1. INTRODUCTION

Given a choice of competing theories, Occam's razor is the principle that directs us to pick the simplest one as the most likely to be correct. This widely held rule of thumb is named after William of Occam (1285-1349), an English philosopher and logician who wrote that "plurality must never be posited without necessity" [1]. Occam asks us to look through a family of possible models and pick the simplest one consistent with some observed data. Consider a mathematical form of Occam's process in which the family of models is a set A of functions on a set X , the simplicity of a model is relative to some partial order on A and the data we are given are the values of some unknown function $f \in A$ on $S \subseteq X$. The key issue is how to decide if and when this data determines a unique function $f \in A$. We make this decision as follows.

Definition 1. *Let (A, \leq) be a partially ordered set of functions on a set X . For $S \subseteq X$ and $f \in A$, we say f is Occam on S or S is Occam for f if f is the least element of $\{g \in A \mid g|_S = f|_S\}$.*

In the case where the partial order is equality, we are familiar with many examples where functions are completely determined by their

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values on particular sets. Entire functions are Occam on any open set. Linear transformations are Occam on any basis. Polynomials of degree at most 3 are Occam on any four element set. Families of functions are often determined by their values on finite sets, so it will be useful to know the size of the smallest set that determines a function uniquely.

Definition 2. Let (A, \leq) be a partially ordered set of functions with domain X and let $f \in A$. The radius of f is defined by $R(f) = \min\{|S| \mid S \subseteq X \text{ and } S \text{ is Occam for } f\}$ if this number is finite and ∞ otherwise. Note that if f is the smallest element of A , then the empty set is Occam for f so $R(f) = 0$.

Here is a simple example that we will shortly generalize in which the radius of each function is equal to 2.

Example 1. Write a function $f: \{1, 2, 3\} \rightarrow \{a, b\}$ as a 3 letter word in the letters from the alphabet $\{a, b\}$ and consider the set of functions represented by $A = \{\underline{a}ba, \underline{b}ab, \underline{a}ab, \underline{b}ba, \underline{a}aa, \underline{b}bb\}$. Then in $(A, =)$ each of these functions is Occam on the subset of $\{1, 2, 3\}$ indicated by the two underscores.

In Example 1, is it possible to make A larger and still have the radius of each function equal to 2? This easily stated question suggests the following unusual family of difficult combinatorial problems.

Problem 1. For fixed m, n and r , what is the maximum possible value of $|A|$ where $|X| = m$, $|Y| = n$ and $(A, =)$ is a set of functions from X to Y with $R(f) \leq r$ for all $f \in A$. Let $\text{Occ}(m, n, r)$ denote this maximum value.

It is straightforward to show that $\text{Occ}(1, n, 1) = n$, $\text{Occ}(m, n, 1) \geq m(n-1)$ and $\text{Occ}(m, n, m) = n^m$. Here is an inequality for the general case.

Theorem 1. $\text{Occ}(m, n, r)$ is less than or equal to the largest number p satisfying the following three conditions for some choice of the non-negative integers x_i :

$$(1) \quad p = \sum_{i=1}^{n^{m-r}} ix_i \quad \sum_{i=1}^{n^{m-r}} x_i \leq n^r \quad p \leq \binom{m}{r} x_1$$

Proof. Let A be any set of functions from an m element set X to an n element set Y . Given an r element subset $S \subseteq X$, define an equivalence relation on A by $f \sim_S g$ if $f|_S = g|_S$. Suppose that the distinct nonempty elements of the partition of A induced by \sim_S are P_1, P_2, \dots, P_k where $k \leq n^r$ since there are at most n^r distinct functions

in the restriction of functions in A to S . There are at most n^{m-r} functions that agree with any particular f on S so $|P_i| \leq n^{m-r}$. If x_i is the number of times that the number i occurs in the list $|P_1|, |P_2|, \dots, |P_k|$, then $|A| = \sum_{i=1}^k |P_i| = \sum_{i=1}^{n^{m-r}} ix_i$. Since $k \leq n^r$, we must also have that $\sum_{i=1}^{n^{m-r}} x_i \leq n^r$. Now suppose that $S_1, S_2, \dots, S_{\binom{m}{r}}$ are all of the r elements subsets of X and let $x_{1,j}$ denote the number of singleton sets in the partition induced by the equivalence relation \sim_{S_j} . By our assumption that every element in A has radius less than or equal to r , we must have $\sum_{j=1}^{\binom{m}{r}} x_{1,j} \geq |A|$. It follows that for at least one value j_0 of j we must have $x_{1,j_0} \geq |A|/\binom{m}{r}$ and the result follows. \square

Theorem 1 can be used in conjunction with a computer to find upper bounds for $Occ(m, n, r)$ for small values of m , n and r .

Computer Search 1. *If we fix $m = 3$ and $r = 2$ and consider the values $n = 2, 3, 4, 5$ in Equation 1, then the corresponding maximum possible values of p are 6, 15, 31, 53 respectively.*

It follows that $Occ(3, 3, 2) \leq 6$, $Occ(3, 3, 2) \leq 15$ and $Occ(3, 4, 2) \leq 31$ but it remains to determine if these upper bounds can actually be attained. Here is what is known about the radius 2 case.

Theorem 2.

- (1) $Occ(3, 2, 2) = 6$
- (2) $Occ(3, 3, 2) = 15$
- (3) $28 \leq Occ(3, 4, 2) \leq 31$
- (4) $4\binom{n}{2} + n \leq Occ(3, n, 2)$
- (5) $2m \leq Occ(m, 2, 2)$

Proof. Parts (1) to (4) of this theorem concern $Occ(3, n, 2)$ which we will bound by generalizing the pattern in Example 1. There are $\binom{n}{2}$ different pairs of letters. For each pair $\{u, v\}$ taken from the n letters in the alphabet, we add the four functions $\underline{uvu}, \underline{vuv}, \underline{uvv}, \underline{vvu}$ for a total of $4\binom{n}{2}$ functions. We then add the n constant functions of the form \underline{uuu} for a total of $4\binom{n}{2} + n$ functions which are all Occam on the 2 underscored positions. It follows that $4\binom{n}{2} + n \leq Occ(3, n, 2)$ and parts (1) to (3) now follow from the results of Computer Search 1.

For Part (5), consider the special case where $m = 5$. The 5 functions $\underline{aaaaa}, \underline{abaaa}, \underline{abbba}, \underline{abbbb}$ and \underline{abbbb} together with the 5 functions $\underline{bbbbb}, \underline{babbb}, \underline{baabb}, \underline{baaab}$ and \underline{baaaa} are all Occam on the 2 underscored positions. This pattern generalizes to give the result. \square

2. THE RADIUS OF A SUBGROUP

In this section, we develop $Occ(G)$ for a group G which is analogous to the rank of G but has the nice property that if H is a subgroup of G then $Occ(H) \leq Occ(G)$. In order to apply the ideas of the last section, we will associate with any group G the set of functions A_G in the following way.

Definition 3. Let G be any group and H be any subgroup of G . Define $\chi_H : G \rightarrow \{0, 1\}$ to be the characteristic function of H that takes the value 1 at each element of H and the value 0 at elements of G not in H . Let A_G denote the set of all characteristic functions of all subgroups of G .

The elements of A_G for the dihedral group D_4 of order 8 are listed on the left hand side Table 1. We now use Definition 2 and define the radius of a subgroup to be the radius of its characteristic function.

Definition 4. Let G be any group and H be any subgroup of G . Define the radius of H in G or $R_G(H)$ to be the radius of χ_H in $(A_G, =)$.

Table 1 shows the radii of the subgroups of D_4 . These numbers capture information about how the subgroup sits inside the group.

The Radius of Subgroups of D_4

χ_H	$R_{D_4}(H)$
11111111	2
10000000	5
10000100	2
10000100	2
10000010	2
10000001	2
10100000	4
11110000	2
10101010	3
10100101	3

TABLE 1. The radii of the subgroups of D_4 found by a computer program.

Theorem 3. If G is any group, then $R_G(G)$ is the rank of G .

Proof. Suppose that rank of G is n and that $S = \{g_1, g_2, \dots, g_n\}$ generates G . S must be Occam for χ_G because if $\chi_H = \chi_G$ on S then

H contains S which implies $H = G$ and therefore $\chi_H = \chi_G$. It follows that $R(\chi_G) \leq n$. Now suppose by way of contradiction that $R(\chi_G) = m < n$ and that $T = \{g_1, g_2, \dots, g_m\}$ is Occam for χ_G . If $H = \langle g_1, g_2, \dots, g_m \rangle$ then $\chi_H = \chi_G$ on T . But H must be a proper subgroup of G which means $\chi_H \neq \chi_G$ and T is not Occam for χ_G . \square

We will be particularly interested in the radius of the identity subgroup of a group G which we will write as $Occ(G)$.

Definition 5. If G is any group, define $Occ(G) = R_G(\{e\})$.

Values of $Occ(G)$ are shown for various groups in Table 2 and we see that these small groups are classified by their order, rank and $Occ(G)$.

The Radius of the Identity for Various Groups

Group G	Order	Rank	$Occ(G) = R_G(\{e\})$
Z_2	2	1	1
Z_3	3	1	1
Z_4	4	1	1
$Z_2 \times Z_2$	4	2	3
Z_5	5	1	1
Z_6	6	1	1
S_3	6	2	4
Z_7	7	1	1
Z_8	8	1	1
Q	8	2	1
$Z_2 \times Z_4$	8	2	3
D_4	8	2	5
Dicyclic	12	2	2
$Z_2 \times Z_6$	12	2	4
A_4	12	2	7
D_6	12	2	8

TABLE 2. The values of $Occ(G)$ found by a computer program.

Unlike the rank, Occ has the nice property that if H is a subgroup of G then $Occ(H) \leq Occ(G)$ as we now show.

Theorem 4. If H is a subgroup of G then $Occ(H) \leq Occ(G)$.

Proof. It is enough to show that if U is Occam for $\chi_{\{e\}}$ in $(A_G, =)$ then $U \cap H$ is Occam for $\chi_{\{e\}}$ in $(A_H, =)$; the relationship between the radii will then follow because $|U \cap H| \leq |U|$. Now U is Occam for $\chi_{\{e\}}$ in $(A_G, =)$ means that if K is any subgroup of G with $U \cap K = U \cap \{e\}$

then $K = \{e\}$. To see that $U \cap H$ is Occam for $\chi_{\{e\}}$ in $(A_H, =)$, suppose that L is any subgroup of H and that $(U \cap H) \cap L = (U \cap H) \cap \{e\}$. This equality simplifies to $U \cap L = U \cap \{e\}$ which implies $L = \{e\}$ since U is Occam for $\chi_{\{e\}}$ in $(A_G, =)$. \square

Using Theorem 4 and Table 2 we can verify the well-known relationships that $Z_2 \times Z_2$ cannot be a subgroup of the Quaterion group of order 8 or the Dicyclic group of order 12 and also that S_3 is not a subgroup of the Dicyclic group.

Questions 1. *What is the smallest example of two nonisomorphic groups with the same order, rank and Occ? How do we characterize the groups G with $\text{Occ}(G) = 1$ which include the cyclic groups and the Quaternion group of order 8 ?*

3. THE FUSION SEQUENCE OF A GROUP

We will make a construction that allows us to associate with any poset of functions (A, \leq) an infinite sequence of positive integers called its fusion sequence. The first step is to identify some special subsets of A .

Definition 6. *Let (A, \leq) be a partially ordered set of functions on a set X and let S be a subset of A . Define the fusion set of S to be the set F_S of all functions $f \in A$ that are Occam on S .*

Give a function f in some poset of functions, we will want to know the size of the smallest fusion set containing f . We call this number the fusion number of f .

Definition 7. *Let (A, \leq) be a partially ordered set of functions on a set X and let $f \in A$. Define the fusion number of f to be $\min\{|F_S| \mid f \in F_S\}$ if finite and infinity otherwise.*

The next construction starts with a poset of functions (A, \leq) and uses its fusion sets to build a new poset of functions on the set A .

Definition 8. *Let (A, \leq) be a partially ordered set of functions on a set X . Define an infinite sequence of partially ordered sets $(A, \leq)_1, (A, \leq)_2, \dots$, called the ascendants of (A, \leq) , as follows. The first ascendent of (A, \leq) is $(A, \leq)_1 = (\{\chi_{F_S} \mid S \subseteq X\}, \prec)$ where \prec orders the characteristic functions by subset inclusion and the domain of each of the functions χ_{F_S} is the set A . Now define $(A, \leq)_2 = (\{\chi_{F_S} \mid S \subseteq X\}, \prec)_1$ and continue inductively.*

We note that χ_A is the maximal element of $(A, \leq)_1$ which implies that every ascendent of (A, \leq) has a maximum element. The radii of

these maximal elements are the terms in the fusion sequence of (A, \leq) that we now define.

Definition 9. Let (A, \leq) be a partially ordered set of functions on a set X . Each ascendent of (A, \leq) has a maximal element so we let F_n be the fusion number of the maximal element of the n th sequent of (A, \leq) . The sequence F_1, F_2, \dots will be called the fusion number sequence of (A, \leq) . If (A, \leq) happens to have a maximal element, then we denote the fusion number of that element by F_0 .

Within this general context of fusion sequences, we now focus on the following poset of functions associated with a group G .

Definition 10. Let G be any group and H be any subgroup of G . Define $\chi_H : G \rightarrow \{0, 1\}$ to be the characteristic function of H that takes the value 1 at each element of H and the value 0 at elements of G not in H . Let (A_G, \prec) denote the set of all characteristic functions of all subgroups of G ordered by inclusion. In other words, $\chi_H \prec \chi_K$ if $H \subseteq K$.

If G is any group then we can give a precise description of the fusion sets.

Theorem 5. Let G be any group. If S is any subset of G and F_S is the fusion set of S with respect to (A_G, \prec) , then $F_S = \{\chi_{\langle R \rangle} \mid R \subseteq S\}$. In other words, F_S consists of the characteristic functions of all subgroups generated by all possible subsets of S with the convention that the empty set generates the identity subgroup.

Proof. To see that $\{\chi_{\langle R \rangle} \mid R \subseteq S\} \subseteq F_S$, let $R \subseteq S$ and suppose that χ_H agrees with $\chi_{\langle R \rangle}$ on S . Then χ_H agrees with $\chi_{\langle R \rangle}$ on R and therefore $R \subseteq H$ and $\langle R \rangle \subseteq H$ so $\chi_{\langle R \rangle}$ is Occam on S . For the reverse inclusion, suppose $\chi_H \subseteq F_S$. Let $R = H \cap S$ which implies $\langle R \rangle \subseteq H$ and $\langle R \rangle \cap S = H \cap S$. It follows that $\chi_{\langle R \rangle} \prec \chi_H$ and that $\chi_{\langle R \rangle}$ agrees with χ_H on S . But by assumption S is Occam for χ_H so we must have $\chi_{\langle R \rangle} = \chi_H$. \square

Table 3 shows the first few terms in the fusion sequences of various groups. Here is a sample calculation for the group $G = Z_4$.

Example 2. Consider the calculation of the fusion sequence of (A_G, \prec) in the case where $G = Z_4 = \{0, 1, 2, 3\}$. In this case,

$A_{Z_4} = \{(1, 0, 0, 0), (1, 0, 1, 0), (1, 1, 1, 1)\}$ and the fusion number of the maximal element $(1, 1, 1, 1)$ is $F_0 = 2$.

$(A_G, \prec)_1 = (\{(1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 1, 1)\}, \prec)$ and the fusion number of the maximal element $(1, 1, 1)$ is $F_1 = 4$.

$(A_G, \prec)_2 = (\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 1, 1, 1)\}, \prec)$ and the fusion number of the maximal element $(1, 1, 1, 1)$ is $F_2 = 2$.

Group	F_0	F_1	F_2	F_3
Z_p p a prime	2	2	2	2
Z_4	2	4	2	4
$Z_2 \times Z_2$	4	8	2	?
Z_6	2	4	4	8
S_3	4	16	2	?
Z_8	2	8	2	8
D_4	4	64	?	?
Q	4	16	2	?
Z_9	2	4	2	4
$Z_3 \times Z_3$	4	16	2	?
Z_{10}	2	4	4	8
Z_{12}	2	8	4	?
Z_{14}	2	4	4	8
Z_{16}	2	16	2	?

TABLE 3. The first four fusion numbers of (A_G, \prec) found with a computer program.

The values of F_0 in Table 3 suggest the following theorem.

Theorem 6. *If G is any group with finite rank n then $F_0 = 2^n$.*

Proof. The maximal element of (A_G, \prec) is χ_G and $F_0 = \min\{|F_S| \mid \chi_G \in F_S\}$ by Definition 8. By Theorem 5, $F_S = \{\chi_{\langle R \rangle} \mid R \subseteq S\}$ which means $F_0 = \min\{|F_S| \mid S \text{ generates } G\}$. Suppose $T = \{g_1, g_2, \dots, g_n\}$ generates G where n is the rank of G . Then every subset of T generates a distinct subgroup of G whose characteristic function is Occam on T and therefore $F_0 \leq |F_T| = \{\chi_{\langle R \rangle} \mid R \subseteq T\} = 2^n$. It remains to show that if S is any generating set for G then $|F_S| \geq 2^n$. For each i , we can find a finite number of elements of S whose product is $g_i \in T$. Together, all of these elements of S used to generate the elements of T form a finite generating set of G which must contain a set U that generates G and is minimal in the sense that no proper subset of U generates G . If R_1 and R_2 are two different subsets of U then we cannot have $\langle R_1 \rangle = \langle R_2 \rangle$ because then at least one of the elements of U could be expressed in terms of other elements of U contradicting U being a minimal generating set. Consequently, the set $\{\langle R \rangle \mid R \subseteq U\}$ must consist of precisely $2^{|U|}$ different subgroups where as usual we take \langle

$\emptyset \geq \{e\}$. Therefore $|F_S| = |\{ \langle R \rangle \mid R \subseteq S \}| \geq |\{ \langle R \rangle \mid R \subseteq U \}| = 2^{|U|} \geq 2^n$, since $|U| \geq n = \text{rank}(G)$ by the definition of rank. \square

It is well known that the rank of a group is not effectively computable in general so it follows from Theorem 6 that the same must be true for F_0 . On almost no evidence we make the following conjecture:

Conjecture 1. *If G is any finite group then the fusion sequence of G is periodic.*

We have used maximal elements and subset inclusion to define the fusion sequence but there are other possible approaches that we will consider elsewhere.

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